JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 107, 156–166 (1985)

Decomposition of the Parametric Space in Multiobjective Convex Programs Using the Generalized Tchebycheff Norm

JERALD P. DAUER

Department of Mathematics and Statistics, University of Nebraska, Lincoln, Nebraska 68588

AND

MOHAMED SAYED ALI OSMAN*

Department of Mathematics, Military Technical College, Cairo, Egypt

Submitted by L. Zadeh

A method for decomposing the parametric space in multiobjective convex programs with or without parameters in the constraints using the generalized Tchebycheff norm is presented. This approach is rather simpler than the corresponding one using the nonnegative weighted sum of objectives. Also, several results are introduced which relate two convex programs with each other, one with parameters in the constraints and the other with parameters in the objective function. Such results make the study of the first type of problems rather simple.
© 1985 Academic Press, Inc.

1. INTRODUCTION

In earlier works Bowman [1] gave necessary and sufficient conditions for the determination of the efficient solutions for multiobjective programs using the generalized Tchebycheff norm (GTN). And in [5], Osman introduced and analyzed the notion of the stability set of the first kind for convex programs with parameters in the right-hand side of the constraints. This paper is devoted to the characterization of such a notion for a certain class of convex programs with parameters in the constraints which results from the scalarization of multiobjective convex programs (MOCP) using the GTN. This characterization enables us to decompose the parametric space for MOCP with or without parameters in the constraints.

* Present address: University of Nebraska, Lincoln, Nebraska 68588.

0022-247X/85 $3.00

The paper also presents several results which relate two convex programs with each other, one with parameters in the constraints and the other with parameters in the objective function.

Two illustrated examples are presented in the paper which clarify the developed theory.

2. Problem Formulation

Let us consider the following vector minimization problem,

$$\begin{gathered} \min F(x), \\ \text{subject to} \\ M = \{x \in R^n / G(x) \leqslant 0\}, \end{gathered} \tag{VMP}$$

where $F\colon R^n \to R^m$, $G\colon R^n \to R^r$ are convex functions of class $C^{(1)}$ on R^n, $F = [f_1 f_2 \cdots f_m]^T$ and $G = [g_1 g_2 \cdots g_r]^T$. Problem (VMP) is assumed to be stable [6].

A point $\bar{x} \in M$ is said to be an efficient solution of (VMP) if there exists no other $x \in M$ such that $F(x) \leqslant F(\bar{x})$ and $F(x) \neq F(\bar{x})$ (see [3]).

A point $\overset{*}{x}$ is said to be a properly efficient solution of (VMP) iff $\overset{*}{x}$ is an efficient solution of (VMP) and there exists a scalar $K > 0$ such that for each i and $x \in M$ satisfying $f_i(x) > f_i(\bar{x})$, we have $f_i(x) - f_i(\bar{x}) \leqslant K(f_j(\bar{x}) - f_j(x))$ for some j such that $f_j(\bar{x}) > f_j(x)$ (see [2, 3]).

Using the generalized Tchebycheff norm defined in [1], a corresponding problem with scalar objective takes the form

$$\begin{gathered} \min \max_i \beta_i |f_i(x) - \overset{*}{u}_i|, \\ \text{subject to } x \in M, \end{gathered} \tag{$\hat{\mathrm{P}}$}$$

where $\beta \in R^m_+$ (the positive orthant of the R^m-space) and $\overset{*}{u} \in R^m$ is an ideal target.

It was shown in [1], that $\bar{x}$ is efficient solution of (VMP) only if it is a solution to $(\hat{\mathrm{P}})$ for some $\beta = \bar{\beta}$, and if the efficient set is uniformly dominant then all solutions to $(\hat{\mathrm{P}})$ are efficient solutions of (VMP).

In this paper $\overset{*}{u}$ will be taken as $\overset{*}{u}_i = \bar{f}_i - \delta$, $i = 1\ 2,..., m$, where $\bar{f}_i = \min_{x \in M} f_i(x)$ and δ is a small positive number and also β will be normalized by the condition $\beta_1 = 1$.

Now, problem $(\hat{\mathrm{P}})$ takes the form

$$\min_{x \in M} \max_i \beta_i [f_i(x) + \delta - \bar{f}_i],$$

or equivalently the form

$$\begin{aligned} &\min z, \\ &\text{subject to} \\ -z+\beta_i[f_i(x)+\delta-\bar{f}_i] &\leqslant 0, \qquad i=1,2,\dots,m, \\ g_k(x) &\leqslant 0, \qquad k=1,2,\dots,r. \end{aligned} \tag{P$_\beta$}$$

The stability of problem $(\mathrm{P})_\beta$ follows directly from the stability of (VMP).

Problem $(\mathrm{P})_\beta$ is a convex programming problem with linear objective and with parameters in the constraints. It is well known from the literature [4] that for any $\beta \in R_+^m$, an optimal solution of problem $(\mathrm{P})_\beta$ cannot be attained at any interior point of its feasible domain.

3. Related Parametric Convex Programs

In this section several results will be presented which relate the convex programming problem $(\mathrm{P})_\beta$ with parameters in the constraints to the convex programming problem $(\mathrm{q})_\lambda$ with parameters in the objective function which is defined as

$$\begin{aligned} &\min \sum_{i=1}^{m} \lambda_i f_i(x), \\ &\text{subject to } x \in M. \end{aligned} \tag{q$_\lambda$}$$

It is well known from the literature that problem $(\mathrm{q})_\lambda$ with $\lambda \geqslant 0, \lambda \neq 0$ can generate all the efficient and the proper efficient solutions of problem (VMP).

At a point $\bar{x}$, the Kuhn–Tucker conditions of problem $(\mathrm{P})_\beta$ takes the form [4]

$$\sum_{i=1}^{m} u_i \beta_i \frac{\partial f_i}{\partial x_\alpha}(\bar{x}) + \sum_{k \in \tau} v_k \frac{\partial g_k}{\partial x_\alpha}(\bar{x}) = 0, \qquad \alpha = 1, 2, \dots, n, \tag{1}$$

$$\sum_{i=1}^{m} u_i = 1, \tag{2}$$

$$-z+\beta_i[f_i(\bar{x})+\delta-\bar{f}_i] \leqslant 0, \qquad i=1,2,\dots,m, \tag{3}$$

$$g_k(\bar{x}) = 0, \qquad k \in \tau \subset \{1, 2, \dots, r\}, \tag{4}$$

$$g_k(\bar{x}) < 0, \qquad k \notin \tau, \tag{5}$$

$$u_i[-z+\beta_i(f_i(\bar{x})+\delta-\bar{f})]=0, \qquad i=1, 2,..., m, \tag{6}$$

$$u_i \geqslant 0, \qquad i=1, 2,..., m, \tag{7}$$

$$v_k \geqslant 0, \qquad k \in \tau. \tag{8}$$

And the Kuhn–Tucker conditions of problem $(\mathrm{q})_\lambda$ takes the form

$$\sum_{i=1}^{m} \lambda_i \frac{\partial f_i}{\partial x_\alpha}(\bar{x}) + \sum_{k \in \tau} v'_k \frac{\partial g_k}{\partial x_\alpha}(\bar{x}) = 0, \qquad \alpha = 1, 2,..., n, \tag{9}$$

$$\begin{aligned} g_k(\bar{x}) &= 0, \qquad k \in \tau, \\ g_k(\bar{x}) &< k \notin \tau, \\ v'_k &\geqslant 0, \qquad k \in \tau. \end{aligned} \tag{10}$$

THEOREM 1. *If either rank*

$$\left[\frac{\partial f_i}{\partial x_\alpha}(\bar{x}) \frac{\partial g_k}{\partial x_\alpha}(\bar{x})\right] = m+s,$$

$i=1, 2,..., m$, $k \in \tau$, $\alpha = 1, 2,..., n$ and s, is the cardinality of the set τ or

$$\frac{\partial f_i}{\partial x_\alpha}(\bar{x}) \geqslant 0 \qquad \text{and} \qquad \frac{\partial g_k}{\partial x_\alpha}(\bar{x}) \geqslant 0$$

for all $i=1, 2,..., m$, $k \in \tau$ and at least one $\alpha \in \{1, 2,..., n\}$, then $\bar{x}$ cannot be an efficient solution of problem (VMP).

Proof. Under our assumptions, system (1) will have the unique solution $u_i=0$, $i=1, 2,..., m$, $v_k=0$, $k \in \tau$, and this contradicts condition (2). Therefore $\bar{x}$ cannot be an optimal solution of problem $(\mathrm{P})_\beta$ for any $\beta > 0$ and hence by Theorem 3 in [1], $\bar{x}$ cannot be an efficient solution of problem (VMP).

Remark 1. Theorem 1 can be proved in the same way using problem $(\mathrm{q})_\lambda$ since the only solution of (9) will be $\lambda_i=0$, $i=1, 2,..., m$, $v'_k=0$, $k \in \tau$.

THEOREM 2. *If $\bar{x}$ is an optimal solution of $(\mathrm{P})_\beta$ for $\beta=\hat{\beta}$ and $(\bar{x}, \bar{z}, \bar{u}, \bar{v})$ solves the Kuhn–Tucker conditions* (1)–(8), *then $\bar{x}$ is an optimal solution of $(\mathrm{q})_\lambda$ for $\lambda = \bar{u}\hat{\beta}$. And if $\bar{x}$ is an optimal solution of $(\mathrm{q})_\lambda$ for $\lambda=\hat{\lambda}$, then $\bar{x}$ is an optimal solution of $(\mathrm{P})_\beta$ for $\beta=\bar{\beta}$, where $f_1(\bar{x})+\delta-\bar{f}_1 = \bar{\beta}_i[f_i(\bar{x})+\delta-\bar{f}_i]$, $i=2, 3,..., m$, $\bar{\beta}_1=1$.*

The proof follows directly from the Kuhn–Tucker conditions of problems $(\mathrm{P})_\beta$ and $(\mathrm{q})_\lambda$.

Remark 2. Theorem 2 shows that problems $(\mathrm{P})_\beta$ and $(\mathrm{q})_\lambda$ generate the same optimal solutions and this makes the study of problems of the type $(\mathrm{P})_\beta$ rather easier by considering the corresponding version of the type $(\mathrm{q})_\lambda$.

THEOREM 3. *If $\bar{x}$ is a proper efficient solution of* (VMP), *then it can be generated from problem* $(\mathrm{P})_\beta$ *with $\beta = \bar{\beta}$, where $\bar{\beta}$ is as defined in Theorem* 2.

Proof. If $\bar{x}$ is a proper efficient solution of (VMP), then there exists $\bar{\lambda} > 0$ such that $\bar{x}$ is an optimal solution of $(\mathrm{q})_{\bar{\lambda}}$. Therefore the results follow directly from Theorem 2.

THEOREM 4. *If $\bar{x}$ is an optimal solution of problem* $(\mathrm{P})_{\overset{*}{\beta}}$ *and* $(\bar{x}, \bar{z}, \bar{u}, \bar{v})$ *solves the Kuhn–Tucker conditions* (1)–(8), *where $\bar{u}_t > 0$ and $f_t(x)$ is a strictly convex function on R^n, then $\bar{x}$ is an efficient solution of* (VMP).

Proof. By the assumptions, it follows that problem $(\mathrm{q})_{\overset{*}{\lambda}}$ has the unique optimal solution $\bar{x}$, where

$$\overset{*}{\lambda}_i = u_i \overset{*}{\beta}_i, \qquad i \neq s$$
$$\overset{*}{\lambda}_s = u_s \overset{*}{\beta}_s > 0$$

(see Theorem 2).

Hence, $\bar{x}$ is an efficient solution of (VMP) (see [3]).

Remark 3. It must be noted that problem $(\mathrm{P})_\beta$ can be written in the equivalent form

$$\min[f_1(x) + \eta + \delta - \bar{f}_1],$$
$$\text{subject to}$$
$$\begin{aligned} \beta_i[f_i(x) + \delta - \bar{f}_i] - f_1(x) - \eta - \delta + \bar{f}_1 &\leqslant 0, \qquad i = 2, 3, \ldots, m, \\ g_k(x) &\leqslant 0, \qquad k = 1, 2, \ldots, r, \\ \eta &\geqslant 0, \end{aligned} \qquad (\overset{*}{\mathrm{P}})_\beta$$

which is obtained by eleminating z from the first constraint of problem $(\mathrm{P})_\beta$.

4. THE STABILITY SET OF THE FIRST KIND

Let $\bar{x}$ be an efficient solution of (VMP), then the stability set of the first kind of problem (VMP) corresponding to $\bar{x}$ which is denoted by $T(\bar{x})$ is defined by

$$T(\bar{x}) = \{\beta \in R^m_+ / \bar{x} \text{ is an efficient solution of (VMP)}\}. \qquad (11)$$

If $\bar{x}$ is an efficient solution of (VMP), then it can be seen that conditions (1), (2), (7) and (8) will put no restrictions on β. Since for any $\beta \in R^m_+$, we can find u satisfying conditions (1), (2) and (8) from the relations

$$\bar{\lambda}_i = k u_i \beta_i, \qquad i = 1, 2, \ldots, m, \ k > 0,$$

$$\sum_{i=1}^{m} u_i = 1,$$

i.e.,

$$u_i = \frac{\bar{\lambda}_i}{\beta_i \sum_{i=1}^{m} (\bar{\lambda}_i / \beta_i)}, \qquad i = 1, 2, \ldots, m, \tag{12}$$

where $\bar{x}$ is an optimal solution of $(\mathrm{q})_{\bar{\lambda}}$.

Therefore, the determination of the set $T(\bar{x})$ depends only on whether any of the variables u_i, $i = 1, 2, \ldots, m$, which solves (1), (2), (7) and (8) is positive or zero.

Let $u_i = 0$, $i \in I \subset \{1, 2, \ldots, m\}$, $u_i > 0$, $i \notin I$ solves (1), (2), (7) and (8), then in order that the other Kuhn–Tucker conditions (3) and (6) are satisfied, we must have

$$z = \beta_i [f_i(\bar{x}) + \delta - \bar{f}_i], \qquad i \notin I,$$

and

$$z \geqslant \beta_i [f_i(\bar{x}) + \delta - \bar{f}_i], \qquad i \in I.$$

Let $D = \{I / u_i = 0,\ i \in I,\ u_i > 0,\ i \notin I$ solves (1), (2), (7) and (8)$\}$, and let

$$T_I(\bar{x}) = \{\beta \in R^m_+ \mid \beta_i [f_i(\bar{x}) + \delta - \bar{f}_i] = \beta_j [f_j(\bar{x}) + \delta - \bar{f}_j],\ i \neq j,\ i, j \notin I,$$
$$\beta_t [f_t(\bar{x}) + \delta - \bar{f}_t] \geqslant \beta_i [f_i(\bar{x}) + \delta - \bar{f}_i],\ i \in I,\ t \notin I\}. \tag{13}$$

Then, it is clear that

$$T(\bar{x}) = \bigcup_{I \in D} T_I(\bar{x}). \tag{14}$$

It must be noted that D is a finite set.

LEMMA 1. *The set* $T_I(\bar{x})$, *which is defined by* (12), *is a polytope and therefore is convex and closed.*

The proof is clear from the definition.

THEOREM 5. *The set $T(\bar{x})$ is closed and star shaped* [5] *with common point of visibility $\beta = \bar{\beta}$, where $\bar{\beta}$ is as defined in Theorem* 2.

Proof. The closedness of $T(\bar{x})$ follows directly from the closedness of $T_I(\bar{x})$ and the finiteness of D (see relation (13)). From Lemma 1 and since $\bar{\beta} \in T_I(\bar{x})$ for all $I \in D$, it follows that $T(\bar{x})$ is a star-shaped set with $\bar{\beta}$ as a point of common visibility.

LEMMA 2. *If $\bar{\lambda}$ solves conditions* (9) *and* (10), *where $\bar{\lambda}_i = 0$, $i \in L \subset \{1, 2,..., m\}$, $\bar{\lambda}_i > 0$, $i \notin L$, then*

$$T_L(\bar{x}) \subset T(\bar{x}).$$

Proof. By assumptions and from (12), it follows that there exists $\overset{*}{u}$ satisfies (1), (2), (7) and (8), where $\overset{*}{u}_i = 0$, $i \in L$, $\bar{u}_i > 0$, $i \notin L$ and hence the result follows directly.

Remark 4. Lemma 2 clarifies the fact that the determination of the set $T(\bar{x})$ is rather simpler than that of the corresponding set using problem $(\mathrm{q})_\lambda$. Since from (9) all the values of λ are required but from (1) and (2) the sign of u is only needed.

Remark 5. The determination of the set $T(\bar{x})$ gives us the possibility of decomposing the parametric space R^m_+ according to the stability sets of the first kind.

LEMMA 3. *If $f_1(x)$ is a strictly convex function on R^n and $f_i(x) - f_1(x)$, $i = 2, 3,..., m$, are convex on the set $\{x \in R^n \mid g_k(x) \leqslant 0,\ k = 1, 2,..., r\}$, then*

$$T(\bar{x}) \cap T(\overset{*}{x}) = \phi, \qquad x \neq \overset{*}{x}.$$

The proof follows from the uniqueness of the optimal solution of $(\overset{*}{\mathrm{P}})_\beta$ which is an equivalent version of $(\mathrm{P})_\beta$.

EXAMPLE 1. Consider the multiobjective convex programming problem

$$\min[x^2 + y^2, x - y, -2x + y],$$
$$\text{subject to} \qquad (\mathrm{VMP})_1$$
$$x + y \leqslant 1, x \geqslant 0, y \geqslant 0.$$

It can be shown that (1, 0) and (0, 1) are two efficient solutions of $(\mathrm{VMP})_1$. To obtain the stability sets of the first kind of problem $(\mathrm{VMP})_1$, we can proceed as described before to deduce the following results

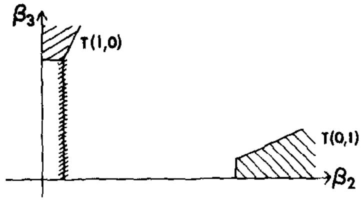

FIGURE 1

(i) For the point $(1, 0)$, we have the possibilities $I_1 = \phi$, $I_2 = \{1\}$, $I_3\{2\}$, $I_4 = \{3\}$ and $I_5 = \{1, 2\}$ with the corresponding sets

$$T_{I_1}(1, 0) = \{\beta \in R^3_+/\beta_1 = 1,\ 1.1 = 2.1\ \beta_2,\ 1.1 = 0.1\ \beta_3\},$$

$$T_{I_2}(1, 0) = \{\beta \in R^3_+/\beta_1 = 1,\ 1.1 \leqslant 2.1\ \beta_2,\ 2.1\ \beta_2 = 0.1\ \beta_3\},$$

$$T_{I_3}(1, 0) = \{\beta \in R^3_+/\beta_1 = 1,\ 2.1\ \beta_2 \leqslant 1.1,\ 1.1 = 0.1\ \beta_3\},$$

$$T_{I_4}(1, 0) = \{\beta \in R^3_+/\beta_1 = 1,\ 1.1 = 2.1\ \beta_2,\ 1.1 \geqslant 0.1\ \beta_3\},$$

$$T_{I_5}(1, 0) = \{\beta \in R^3_+/\beta_1 = 1,\ 1.1 \leqslant 0.1\ \beta_3,\ 2.1\ \beta_2 \leqslant 0.1\ \beta_3\},$$

where δ is taken equal to 0.1.

Therefore, $T(1, 0) = \bigcup_{i=1}^{5} T_{I_i}(1, 0)$ (see Fig. 1).

(ii) For the point $(0, 1)$, we have the possibilities $J_1 = \phi$, $J_2 = \{1\}$, $J_3 = \{3\}$, $J_4 = \{1, 3\}$, with the corresponding sets

$$T_{J_1}(0, 1) = \{\beta \in R^3_+/\beta_1 = 1,\ 1.1 = 0.1\ \beta_2,\ 1.1 = 3.1\ \beta_3\},$$

$$T_{J_2}(0, 1) = \{\beta \in R^3_+/\beta_1 = 1,\ 1.1 \leqslant 0.1\ \beta_2,\ 0.1\ \beta_2 = 3.1\ \beta_3\},$$

$$T_{J_3}(0, 1) = \{\beta \in R^3_+/\beta_1 = 1,\ 1.1 = 0.1\ \beta_2,\ 1.1 \geqslant 3.1\ \beta_3\},$$

$$T_{J_4}(0, 1) = \{\beta \in R^3_+/\beta_1 = 1,\ 1.1 \leqslant 0.1\ \beta_2,\ 0.1\ \beta_2 \geqslant 3.1\ \beta_3\},$$

where δ is taken equal to 0.1.

Therefore, $T(1, 0) = \bigcup_{i=1}^{4} T_{J_i}(1.0)$ (see Fig. 1).

5. MULTIOBJECTIVE CONVEX PROGRAMS WITH PARAMETERS IN THE CONSTRAINTS

Let us consider the following multiobjective convex program with parameters in the right hand side of the constraints

$$\min F(x),$$

$$\text{subject to} \qquad (\text{VMP})_v$$

$$M(v) = \{x \in R^n \mid G(x) \leqslant v\},$$

where $v = [v_1 v_2 \cdots v_r]^T$ is any vector in R^r. Folowing the same steps as before, the corresponding problem with single objective using the generalized Tchebycheff norm takes the form

$$\begin{aligned} &\min z, \\ &\text{subject to} \\ &-z + \beta_i[f_i(x) + \delta - \bar{f}_i] \leqslant 0, \; i = 1, 2, \ldots, m, \\ &g_k(x) \leqslant v_k, \; k = 1, 2, \ldots, r, \end{aligned} \qquad (\mathrm{P})_{v,\beta}$$

which is a convex programming problem with parameters in the constraints.

Remark 6. It is clear that problem $(\mathrm{P})_{v,\beta}$ has a simpler nature than problem $(\mathrm{q})_{v,\lambda}$ which can be formulated using the usual scalarization procedure (see problem $(\mathrm{q})_\lambda$) since the latter will have parameters in both the objective function and the constraints.

Suppose that for $(\overset{*}{v}, \overset{*}{\beta}) \in R^T \times R^m_+$ an efficient solution of $(\mathrm{VMP})_v$ is found to be $\bar{x}$, then the stability set of the first kind of problem $(\mathrm{VMP})_v$ corresponding to $\bar{x}$ which is denoted by $T'(\bar{x})$ is defined by

$$T'(\bar{x}) = \{(v, \beta) \in R^r \times R^m_+ \mid \bar{x} \text{ is an efficient solution of } (\mathrm{VMP})_v\}. \qquad (15)$$

At an efficient solution $\bar{x}$ of $(\mathrm{VMP})_v$, the Kuhn–Tucker conditions of problem $(\mathrm{P})_{v,\beta}$ take the form

$$\begin{aligned} &\sum_{i=1}^m u_i \beta_i \frac{\partial f_i}{\partial x_\alpha}(\bar{x}) + \sum_{k=1}^r v_k \frac{\partial g_k}{\partial x_\alpha}(\bar{x}) = 0, \qquad \alpha = 1, 2, \ldots, n, \\ &\sum_{i=1}^m u_i = 1, \end{aligned} \qquad (16)$$

$$\begin{aligned} -z + \beta_i[f_i(\bar{x}) + \delta - \bar{f}_i] &\leqslant 0, & i &= 1, 2, \ldots, m, \\ g_k(\bar{x}) &\leqslant v_k, & k &= 1, 2, \ldots, r, \end{aligned} \qquad (17)$$

$$\begin{aligned} u_i[-z + \beta_i(f_i(\bar{x}) + \delta - \bar{f}_i)] &= 0, & i &= 1, 2, \ldots, m, \\ v_k(g_k(\bar{x}) - v_k) &= 0, & k &= 1\ 2, \ldots, r, \\ u_i &\geqslant 0, & i &= 1, 2, \ldots, m, \\ v_k &\geqslant 0, & k &= 1, 2, \ldots, r. \end{aligned} \qquad (18)$$

To determine the set $T'(\bar{x})$, we must determine at first whether any of the variables u_i, $i = 1, 2, \ldots, m$ and any of the variables v_k, $k = 1, 2, \ldots, r$, which solve (2), (7), (8), (16) are zero or positive.

Let

$$u_i = 0,\ i \in I \subset \{1, 2, ..., m\}, \qquad u_i > 0,\ i \notin I,$$
$$v_k = 0,\ k \in J \subset \{1, 2, ..., r\}, \qquad v_k > 0,\ k \in J$$

solve (2), (7), (8), (16) then in order that the other Kuhn–Tucker conditions of problem $(\mathrm{P})_{v,\beta}$ are satisfied, we must have

$$\begin{aligned} z &= \beta_i[f_i(\bar{x}) + \delta - \bar{f}_i], && i \notin I, \\ z &\geqslant \beta_i[f_i(\bar{x}) + \delta - \bar{f}_i], && i \in I, \\ v_k &= g_k(\bar{x}), && k \notin J, \\ v_k &\geqslant g_k(\bar{x}), && k \in J. \end{aligned}$$

Let $D' = \{(I, J)/u_i = 0,\ i \in I,\ u_i > 0,\ i \notin I,\ v_k = 0,\ k \in J,$

$v_k > 0,\ k \notin J$ Solve (2), (7), (8) and (16)$\}$,

and let

$$\begin{aligned} T'_{I,J}(\bar{x}) = \{(v, \beta) \in R^r \times R^m_+ / v_k = g_k(\bar{x}),\ & k \notin J, \\ v_k \geqslant g_k(\bar{x}),\ & k \in J, \\ \beta \in T_I(\bar{x})\}. & \end{aligned} \tag{19}$$

Then, it is clear that

$$T'(\bar{x}) = \bigcup_{(I,J) \in D'} T_{I,J}(\bar{x}). \tag{20}$$

The set D' is clearly finite.

THEOREM 6. *The set $T'(\bar{x})$ is closed, star shaped with the point $(\bar{v}, \bar{\beta})$ as a point of common visibility, where $\bar{v} = G(\bar{x})$ and $\bar{\beta}$ as defined in Theorem* 2.

The proof can be done in a similar way as that for Theorem 5.

EXAMPLE 2. Consider the following multiobjective program with a parameter in the right hand side of one constraint

$$\min[x^2 + y^2, x^2 - y],$$
$$\text{subject to}$$
$$x + y \leqslant v,$$
$$x \geqslant 0,\ y \geqslant 0.$$

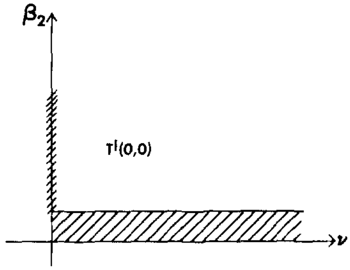

FIGURE 2

For $v=1$, an efficient solution is found to be $(x, y)=(0, 0)$. To determine the stability set of the first kind of this problem corresponding to $(0, 0)$, we will have the following possibilities.

$$(I_1, J_1)=\phi,\ (I_2, J_2)=(\{1\}, \phi),\ (I_3, J_3)=(\{2\}, \phi),$$
$$(I_4, J_4)=(\{2\}, \{1\}),$$

with the corresponding sets

$$T'_{I_1,J_1}(0, 0)=\{(v, \beta)\in R\times R^2_+ \mid v=0, \beta_1=1, 0.1=1.1\ \beta_2\},$$
$$T'_{I_2,J_2}(0, 0)=\{(v, \beta)\in R\times R^2_+ \mid v=0, \beta_1=1, 0.1\leqslant 1.1\ \beta_2\},$$
$$T'_{I_3,J_3}(0, 0)=\{(v, \beta)\in R\times R^2_+ \mid v=0, \beta_1=1, 0.1\geqslant 1.1\ \beta_2\},$$
$$T'_{I_4,J_4}(0, 0)=\{(v, \beta)\in R\times R^2_+ \mid v\geqslant 0, \beta_1=1, 0.1\geqslant 1.1\ \beta_2\},$$

where we take $\delta=0.1$.

Therefore, $T'(0, 0)=\bigcup_{i=1}^4 T'_{I_i,J_i}(0, 0)$ (see Fig. 2).

REFERENCES

1. V. J. BOWMAN JR., On the relationship of the Tchebycheff norm and the efficient frontier of multiple-criteria objectives, *in* "Multiple Criteria Decision Making" (H. Thiriez and S. Zionts, Eds), pp. 76–85, Berlin/New York/Heidelberg, 1976.
2. A. M. GEOFFRION, Proper efficiency and the theory of vector maximization, *J. Math. Anal. Appl.* **22** (1968), 618–630.
3. C. L. HWANG AND A. S. MASUD, "Multiple Objective Decision-Making Methods and Applications: A State of the Art Survey," Springer-Verlag, Berlin/New York, 1979.
4. O. MANGASARIAN, "Nonlinear Programming," McGraw–Hill, New York/London, 1964.
5. M. S. A. OSMAN, Qualitative analysis of basic notions in parametric convex programming, I (parameters in the constraints), *Apl. Mat.* **22** (1977), 318–332.
6. R. T. ROCKAFELLAR, Duality and stability in extremum problems involving convex functions, *Pacific J. Math.* **21** (1967), 167–187.